Social Network Analysis in Python

Enrico Franchi (efranchi@ce.unipr.it)
Dipartimento Ingegneria dell'Informazione
Università degli Studi di Parma

Outline

- Introduction
- Data representation
- Network Properties
  - Network Level
  - Group Level
  - Node Level
- Visualization
- PageRank

Social Network Analysis in Python

Social Network Analysis in Python

"A **social network** is a finite set of actors and the relations defined on them"

"A **social network** is a finite set of actors and the relations defined on them"

NETWORK

"A **social network** is a finite set of actors and the relations defined on them"

NETWORK

PEOPLE [ACTORS]

"A **social network** is a finite set of actors and the relations defined on them"

NETWORK

PEOPLE [ACTORS]

CONNECTIONS [RELATIONS]

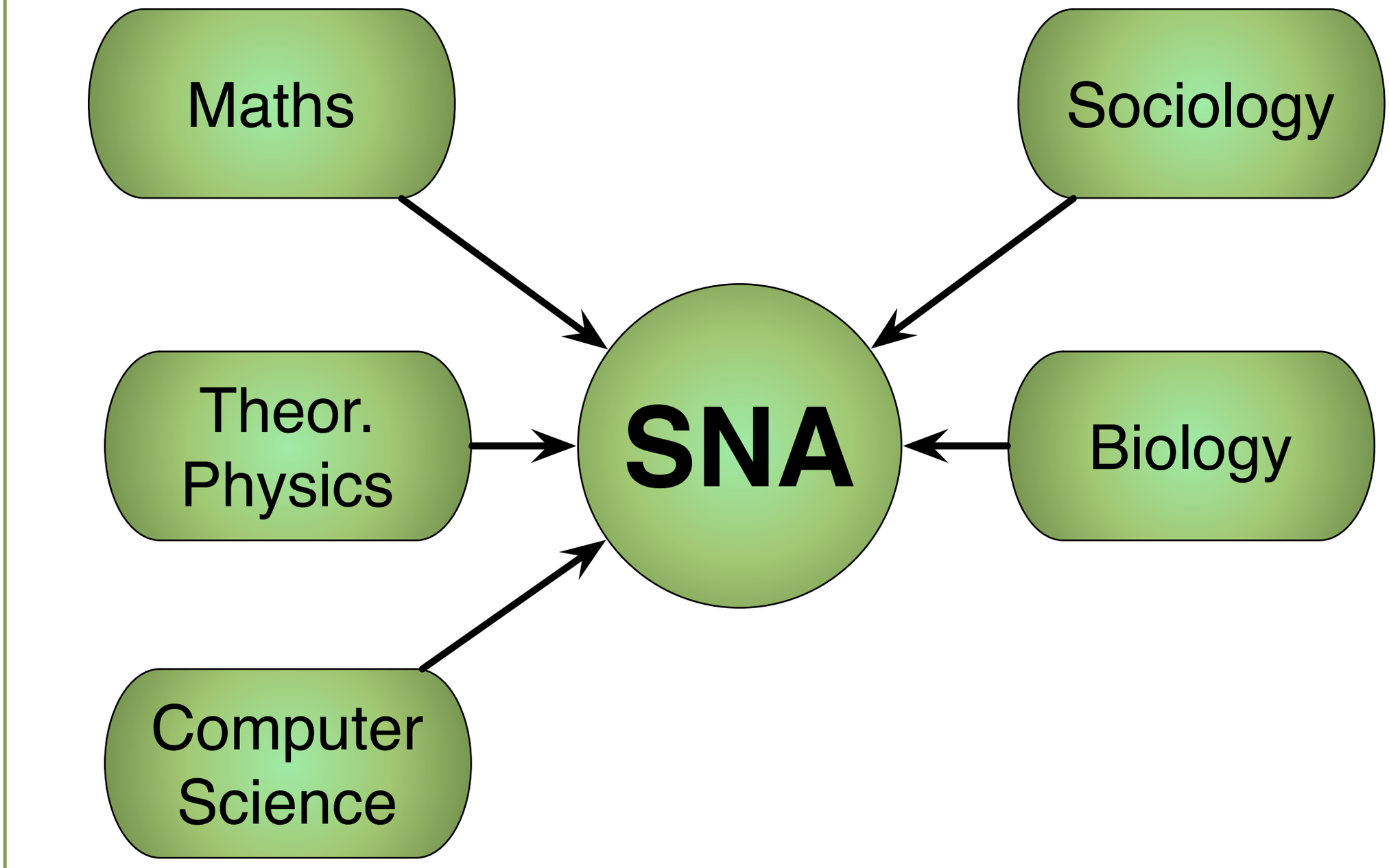
Social Network Analysis in Python

Social Network Analysis in Python

Social Network Analysis analyzes the *structure of relations among (social, political, organizational) actors*

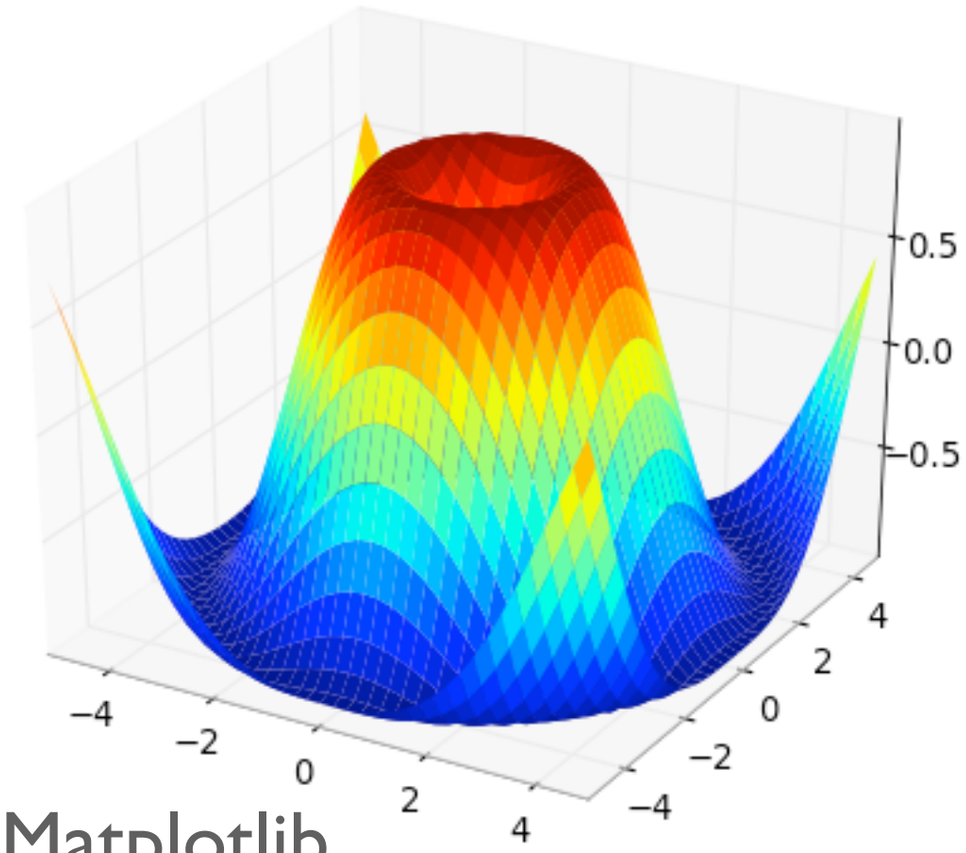
Social Network Analysis

- Complex Network Analysis
  - Element Level
  - Group Level
  - Network Level
- Complex Network Models
  - Data-driven Approach
  - Mechanistic Approach
  - Game-Theoretic Approach
- Processes on Complex Networks


Maths
Sociology
Theor.
Physics
SNA
Biology
Computer
Science

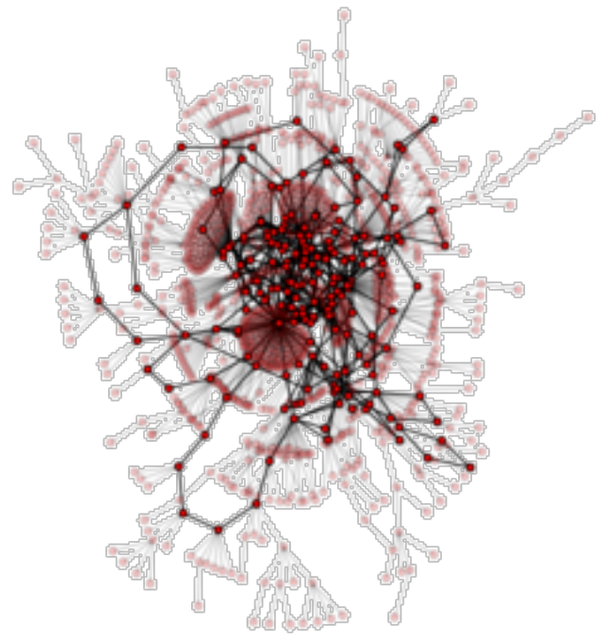
Social Network Analysis in Python

Social Network Analysis in Python



Matplotlib

Gephi
makes graphs handy



Networkx

SciPy.org

NumPy

IP[y]:

GUESS The Graph Exploration System

Dimensions of Social Networks

	Nodes	Edges	Max Nodes	Density
student relationships	6E+02	5E+02	3.28E+05	1.45E-03
physics co-authorship	5E+04	2E+05	2.80E+09	8.76E-05
math co-authorship	3E+05	5E+05	6.42E+10	7.74E-06
film actors	4E+05	3E+07	2.02E+11	1.26E-04
Skype instantaneous usage	*2E+07*	3E+06	4.00E+14	7.50E-09
telephone call graph	5E+07	8E+07	2.21E+15	3.62E-08
World "Friendship"	7E+09	1E+12	4.76E+19	2.17E-08

Density: m/n^2

Social Networks are almost always sparse

Dimensions of Social Networks

	Nodes	Edges	Max Nodes	Density
student relationships	6E+02	5E+02	3.28E+05	1.45E-03
physics co-authorship	5E+04	2E+05	2.80E+09	8.76E-05
math co-authorship	3E+05	5E+05	6.42E+10	7.74E-06
film actors	4E+05	3E+07	2.02E+11	1.26E-04
Skype instantaneous usage	*2E+07*	3E+06	4.00E+14	7.50E-09
telephone call graph	5E+07	8E+07	2.21E+15	3.62E-08
World "Friendship"	7E+09	1E+12	4.76E+19	2.17E-08

Density: m/n^2

n · dunbar number

Social Networks are almost always sparse

Data Representation

Mathematical Representations

- A graph G is a triple $G=(V, E, e)$ where V is a set of vertices, E is a set of edges and e is a function $e: E \rightarrow V \times V$ mapping edges to their endpoints
- Sometimes is useful to consider $E = V \times V$
- Graphs can have self-links, multiple links (multi-graph), labelled links
- A graph is directed if $e: E \rightarrow V^{(2)}$
- We indicate with n the order $|V|$ of the graph and with m the size $|E|$ of the graph

Computer Representations

- Adjacency Matrix (n×n): $A_{ij} = \begin{cases} 1 \text{ if (i,j)} \in \text{img}(e) \\ 0 \text{ otherwise} \end{cases}$
- Incidence Matrix (n×m): $B_{vj} = \begin{cases} 1 \text{ if } (u,j)\ \exists v\,(u,v) = e(j) \\ 0 \text{ otherwise} \end{cases}$

- Adjacency List:
  - dict: keys are nodes, values are lists/sets of nodes
- Incidence List:
  - dict: keys are nodes, values are sequences of edges
  - dict: keys are nodes, values are the endpoints tuples

Sparse Matrices

- An adjacency/incidence matrix is better represented with scipy.sparse matrices
  - Different implementations provide different trade-offs
  - Sometimes it is possible to convert matrices in different formats efficiently
  - Different implementations have different points of strengths (choose the appropriate implementation depending on what is needed)
- numpy matrices are great, but only for small networks
- Relatively easy to write some algorithms and efficiency depends from the implementation (and C code)
- Cumbersome to store additional data on nodes or edges

Incidence List

- This is how graphs are represented in Jung (a widespread Java library which can be used with Jython)
- Egde objects are “reified” (contain attributes)
- Node objects usually contain attributes as well
- “Very OO” [perhaps an overkill]
- Following the definition leads to inefficiencies

```
class IncidenceListGraph(object):
    def __init__(self):
        self.incidence = {}
        self.endpoints = {}

    def add_node(self, node):
        self.incidence.setdefault(node, set())

    def add_edge(self, edge, start, end):
        self.endpoints[edge] = (start, end)
        try:
            starting_node_links = self.incidence[start]
            end_node_links = self.incidence[end]
        except KeyError:
            return False
        else:
            starting_node_links.add(edge)
            end_node_links.add(edge)
            return True
```

Slow lookup: is there a connection between *i* and *j*?

```
class IncidenceListJUNGGraph(object):
    def __init__(self):
        self.incidence = {}
        self.endpoints = {}

    def add_node(self, node):
        self.incidence.setdefault(node, dict())

    def add_edge(self, edge, start, end):
        self.endpoints[edge] = (start, end)
        try:
            starting_node_links = self.incidence[start]
            end_node_links = self.incidence[end]
        except KeyError:
            return False
        else:
            starting_node_links[end] = edge
            end_node_links[start] = edge
            return True
```

Adjacency List

- Somewhat the "more pythonic way" (http://www.python.org/doc/essays/graphs.html)
- Rather efficient in terms of space and costs of elementary operations
- Networkx implementation of graphs is based on this idea

```
graph = {'A': ['B', 'C'],
         'B': ['C', 'D'],
         'C': ['D'],
         'D': ['C'],
         'E': ['F'],
         'F': ['C']}
```

```
class AdjacencyListGraph(object):
    def __init__(self):
        self.node = {}
        self.adj = {}

    def add_node(self, node, **attrs):
        if node not in self.adj:
            self.adj[node] = {}
            self.node[node] = attrs
        else: # update attr even if node already exists
            self.node[node].update(attrs)

    def add_edge(self, u, v, **attrs):
        if u not in self.adj:
            self.adj[u] = {}
            self.node[u] = {}
        if v not in self.adj:
            self.adj[v] = {}
            self.node[v] = {}

        datadict=self.adj[u].get(v,{})
        datadict.update(attrs)

        self.adj[u][v] = datadict
        self.adj[v][u] = datadict
```

Counting edges is not efficient!

Graph & File Formats

- Networkx graphs can be created from and converted to
  - numpy matrices
  - scipy sparse matrices
  - dicts of lists
  - dicts of dicts
  - lists of edges
- Networkx graphs can be read from and saved to the following formats
  - textual formats (adj lists)
  - GEXF (gephi)
  - GML
  - GraphML
  - Pajek
  - ...

Network Properties

Degree

- Directed
  - indegree: $k_i^{\text{in}} = \sum_j A_{ji}$
  - outdegree: $k_i^{\text{out}} = \sum_j A_{ij}$
  - degree: $k_i = k_i^{\text{in}} + k_i^{\text{out}}$
- Undirected
  - degree: $k_i = \sum_j A_{ji} = \sum_j A_{ij}$

- Mean degree: $\bar{k} = \frac{1}{n}\sum_i k_i$
- Degree distribution: $p_k = \frac{1}{n}\#\{i | k_i = k\}$

Network Level Properties

- Characteristic Path Length
- Clustering Coefficient
- Degree Distribution

- Distribution of other node level properties
- Correlations of node level properties
  - Assortativity (epidemics)

Characteristic Path Length

- $L(i,j)$ is the length shortest path(s) between i and j
- $L_i = (n-1)^{-1} \sum_j L(i,j)$ is the average shortest path of i
- $\bar{L} = n^{-1} \sum_i L_i$ is the characteristic path length of the network (CPL)
- Computation of all the shortest paths is usually done with Dijkstra algorithm (networkx)
  - In practice: $O(nm + n^2 \log n)$
- Networkx can compute shortest paths, CPL, etc.

```python
from heapq import heappush, heappop
# based on recipe 119466
def dijkstra_shortest_path(graph, source):
    distances = {}
    predecessors = {}
    seen = {source: 0}
    priority_queue = [(0, source)]

    while priority_queue:
        v_dist, v = heappop(priority_queue)
        distances[v] = v_dist

        for w in graph[v]:
            vw_dist = distances[v] + 1
            if w not in seen or vw_dist < seen[w]:
                seen[w] = vw_dist
                heappush(priority_queue,(vw_dist,w))
                predecessors[w] = v

    return distances, predecessors
```

$O(m \cdot \text{push}_Q + n \cdot \text{ex-min}_Q) = O(m \log n + n \log n)$

```python
from heapq import heappush, heappop
# based on recipe 119466
def dijkstra_shortest_path(graph, source):
    distances = {}
    predecessors = {}
    seen = {source: 0}
    priority_queue = [(0, source)]

    while priority_queue:
        v_dist, v = heappop(priority_queue)
        distances[v] = v_dist

        for w in graph[v]:
            vw_dist = distances[v] + 1
            if w not in seen or vw_dist < seen[w]:
                seen[w] = vw_dist
                heappush(priority_queue,(vw_dist,w))
                predecessors[w] = v

    return distances, predecessors
```

$O(m \cdot \text{push}_Q + n \cdot \text{ex-min}_Q) = O(m \log n + n \log n)$

```
from heapq import heappush, heappop
# based on recipe 119466
def dijkstra_shortest_path(graph, source):
    distances = {}
    predecessors = {}
    seen = {source: 0}
    priority_queue = [(0, source)]

    while priority_queue:
        v_dist, v = heappop(priority_queue)
        distances[v] = v_dist

        for w in graph[v]:
            vw_dist = distances[v] + 1
            if w not in seen or vw_dist < seen[w]:
                seen[w] = vw_dist
                heappush(priority_queue,(vw_dist,w))
                predecessors[w] = v

    return distances, predecessors
```

$O(m \cdot \text{push}_Q + n \cdot \text{ex-min}_Q) = O(m \log n + n \log n)$

From mean to median

- Computing the the shortest paths for all but the smallest networks (< 1000 nodes) is essentially not feasible

- However, the *median* of the average shortest paths is easier to estimate and is a good metric, thus it is common to define the characteristic path length as the *median* (instead of the *mean*) of the average shortest path length

Approximate Medians

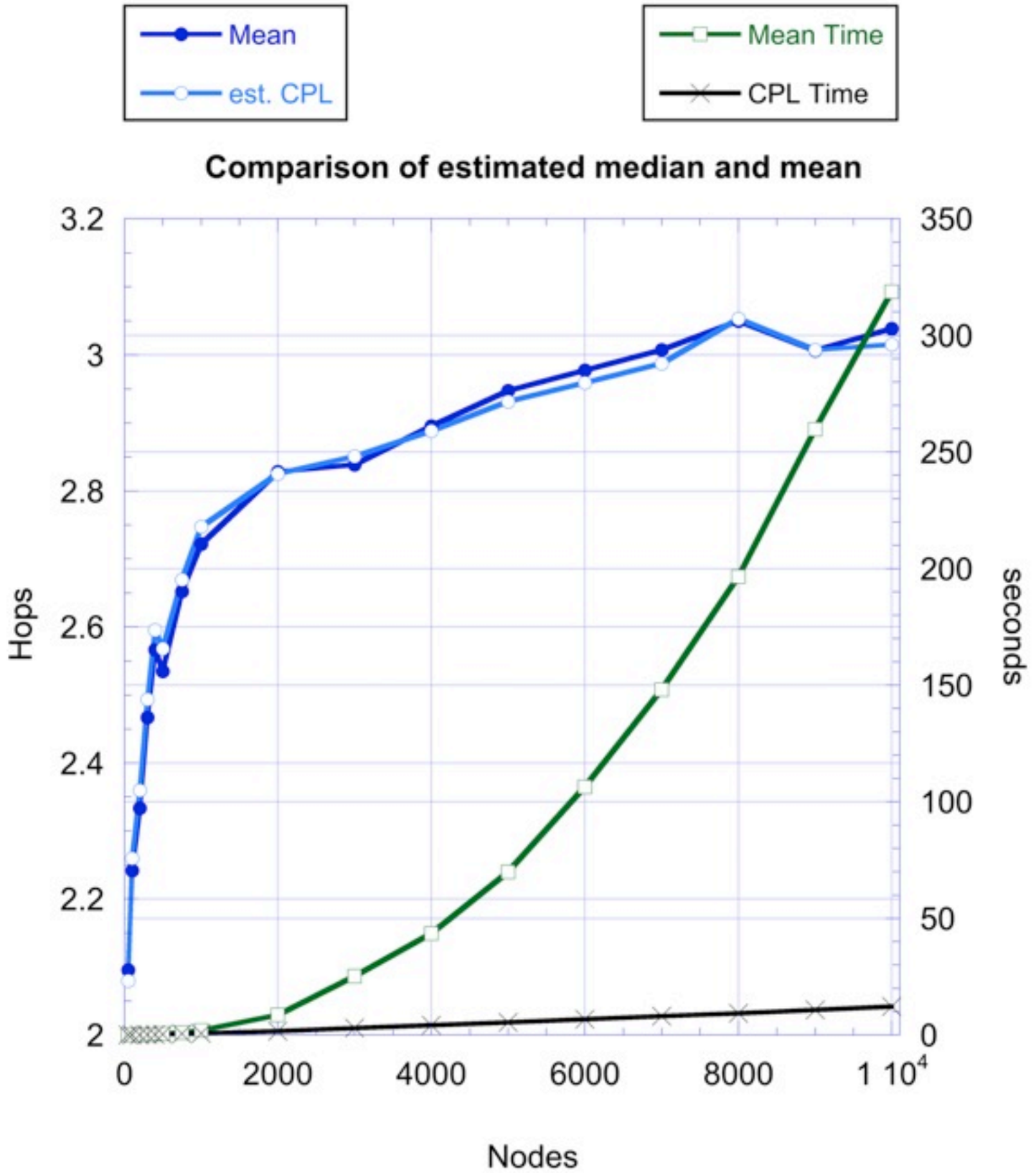
- $M_{(q)}$ is a q-median if at least qn of the numbers in a set are less than or equal to $M_{(q)}$ and at least $(1\text{-}q)n$ are greater than $M_{(q)}$
  - So a regular median is a 0.5-median
- $L_{(q, \delta)}$ is a (q, δ)-median if at least $qn(1\text{-}\delta)$ elements in the set are less than or equal to $L_{(q, \delta)}$ and at least $(1\text{-}q)n(1\text{-}\delta)$ are greater than $L_{(q, \delta)}$

Huber Algorithm

- A value for $L_{(q,\delta)}$ can be found taking a sample of s elements and looking at the $M_{(q)}$ median

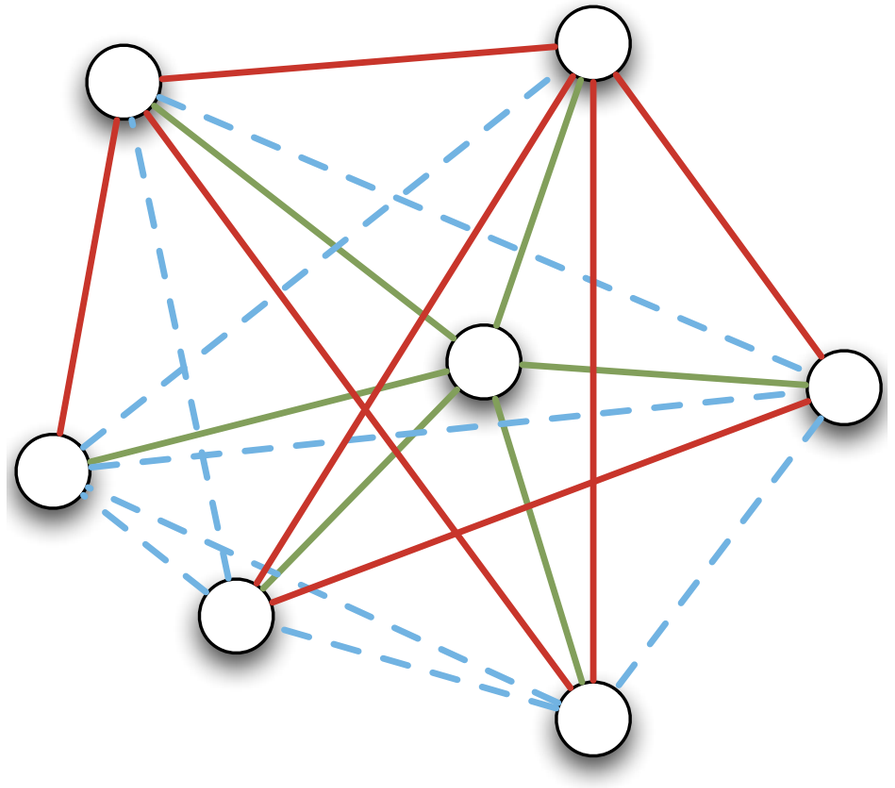
- If $s = \frac{2}{q^2} \ln \frac{2}{\epsilon} \frac{(1-\delta)^2}{\delta^2}$ the value is correct with probability $1-\epsilon$

```
def approximate_cpl(graph, q=0.5, delta=0.15, eps=0.05):
    s = estimate_s(q, delta, eps)
    s = int(math.ceil(s))
    if graph.number_of_nodes() <= s:
        sample = graph.nodes_iter()
    else:
        sample = random.sample(graph.adj.keys(), s)

    averages = []
    for node in sample:
        path_lengths =
nx.single_source_shortest_path_length(graph, node)
        average = sum(path_lengths.itervalues())/float
(len(path_lengths))
        averages.append(average)
    averages.sort()
    median_index = int(len(averages) * q + 1)
    return averages[median_index]
```


Mean
est. CPL
Mean Time
CPL Time
Comparison of estimated median and mean
3.2
3
2.8
2.6
2.4
2.2
2
Hops
350
300
250
200
150
100
50
0
seconds
0
2000
4000
6000
8000
1 10^4
Nodes

Local Clustering Coefficient

- Green lines are the links between the *i* and its neighbors
- Red lines are the links between the neighbors of *i*
- Cyan dotted lines are in the complete graph and not in the network
- C_i=7/15=0.46



Local Clustering Coefficient

- Let T(i) the number of distinct triangles having node *i* as a vertex
- The maximum number of possible connections in the neighborhood of *i* is $k_i(k_i-1)/2$
- The *local clustering coefficient* of *i* is: $C_i = \binom{k_i}{2}^{-1} T(i) = \frac{2T(i)}{k_i(k_i-1)}$
- The clustering coefficient is:

$$C = \frac{1}{n}\sum_{i \in V} C_i$$

- A different (and better) definition exists:

$$C = \frac{(\text{number of closed paths of length 2})}{(\text{number of paths of length 2})} = \frac{(\text{number of triangles}) \times 3}{(\text{number of connected triples})}$$

Local Clustering Coefficient

- Let T(i) the number of distinct triangles having node *i* as a vertex
- The maximum number of possible connections in the neighborhood of *i* is $k_i(k_i-1)/2$
- The *local clustering coefficient* of *i* is:
- The clustering coefficient is:

$$C = \frac{1}{n}\sum_{i \in V} C_i$$

- A different (and better) definition exists:

$$C = \frac{(\text{number of closed paths of length 2})}{(\text{number of paths of length 2})} = \frac{(\text{number of triangles}) \times 3}{(\text{number of connected triples})}$$

nx.clustering

$$C_i = \binom{k_i}{2}^{-1} T(i) = \frac{2T(i)}{k_i(k_i-1)}$$

Local Clustering Coefficient

- Let T(i) the number of distinct triangles having node *i* as a vertex
- The maximum number of possible connections in the neighborhood of *i* is $k_i(k_i-1)/2$
- The *local clustering coefficient* of *i* is: $C_i = \binom{k_i}{2}^{-1} T(i) = \frac{2T(i)}{k_i(k_i-1)}$
- The clustering coefficient is:

$$C = \frac{1}{n}\sum_{i \in V} C_i$$

nx.average_clustering

- A different (and better) definition exists:

$$C = \frac{(\text{number of closed paths of length 2})}{(\text{number of paths of length 2})} = \frac{(\text{number of triangles}) \times 3}{(\text{number of connected triples})}$$

Local Clustering Coefficient

- Let T(i) the number of distinct triangles having node *i* as a vertex
- The maximum number of possible connections in the neighborhood of *i* is $k_i(k_i-1)/2$
- The *local clustering coefficient* of *i* is: $C_i = \binom{k_i}{2}^{-1} T(i) = \frac{2T(i)}{k_i(k_i-1)}$
- The clustering coefficient is:

$$C = \frac{1}{n}\sum_{i \in V} C_i$$

- A different (and better) definition exists:

nx.transitivity

$$C = \frac{(\text{number of closed paths of length 2})}{(\text{number of paths of length 2})} = \frac{(\text{number of triangles}) \times 3}{(\text{number of connected triples})}$$

Local Clustering Coefficient

- Let T(i) the number of distinct triangles having node *i* as a vertex
- The maximum number of possible connections in the neighborhood of *i* is $k_i(k_i-1)/2$
- The *local clustering coefficient* of *i* is: $C_i = \binom{k_i}{2}^{-1} T(i) = \frac{2T(i)}{k_i(k_i-1)}$
- The clustering coefficient is:

$$C = \frac{1}{n}\sum_{i \in V} C_i$$

- A different (and better) definition exists:

$$C = \frac{(\text{number of closed paths of length 2})}{(\text{number of paths of length 2})} = \frac{(\text{number of triangles}) \times 3}{(\text{number of connected triples})}$$

Degree Distribution

- Degree distribution: frequency of the degrees of the nodes
- Most social networks have right-skewed degree distributions
  - Most nodes have low degree, some have exceptionally high degree
- Keep in mind when sampling
  - node based sampling
  - edge based sampling

Power-Laws

- General form of a power-law degree distribution

$$\ln p_k = -\alpha \ln k + c$$

$$p_k = Ck^{-\alpha}$$

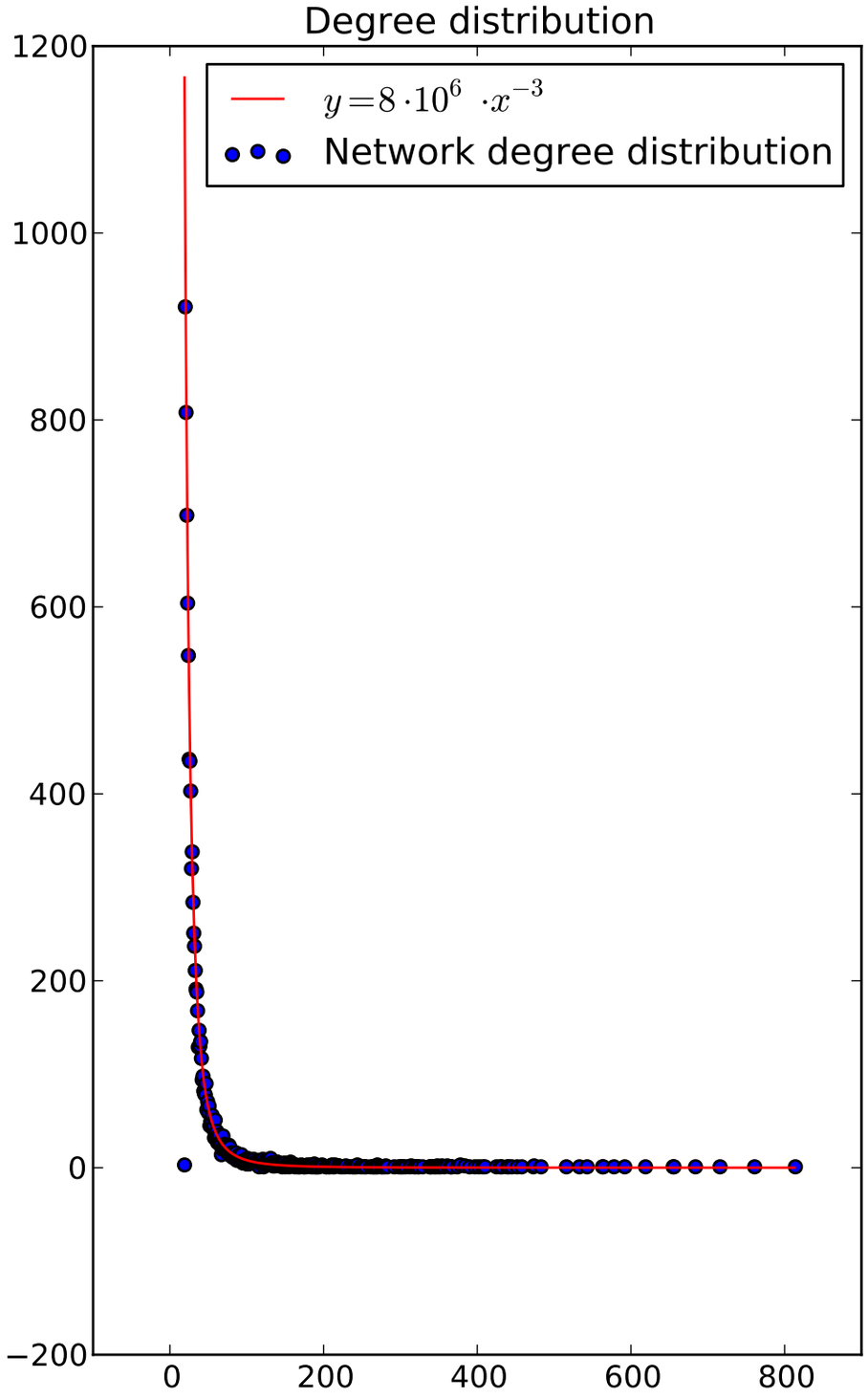
- Graphs with power law degree distribution are called *scale-free*

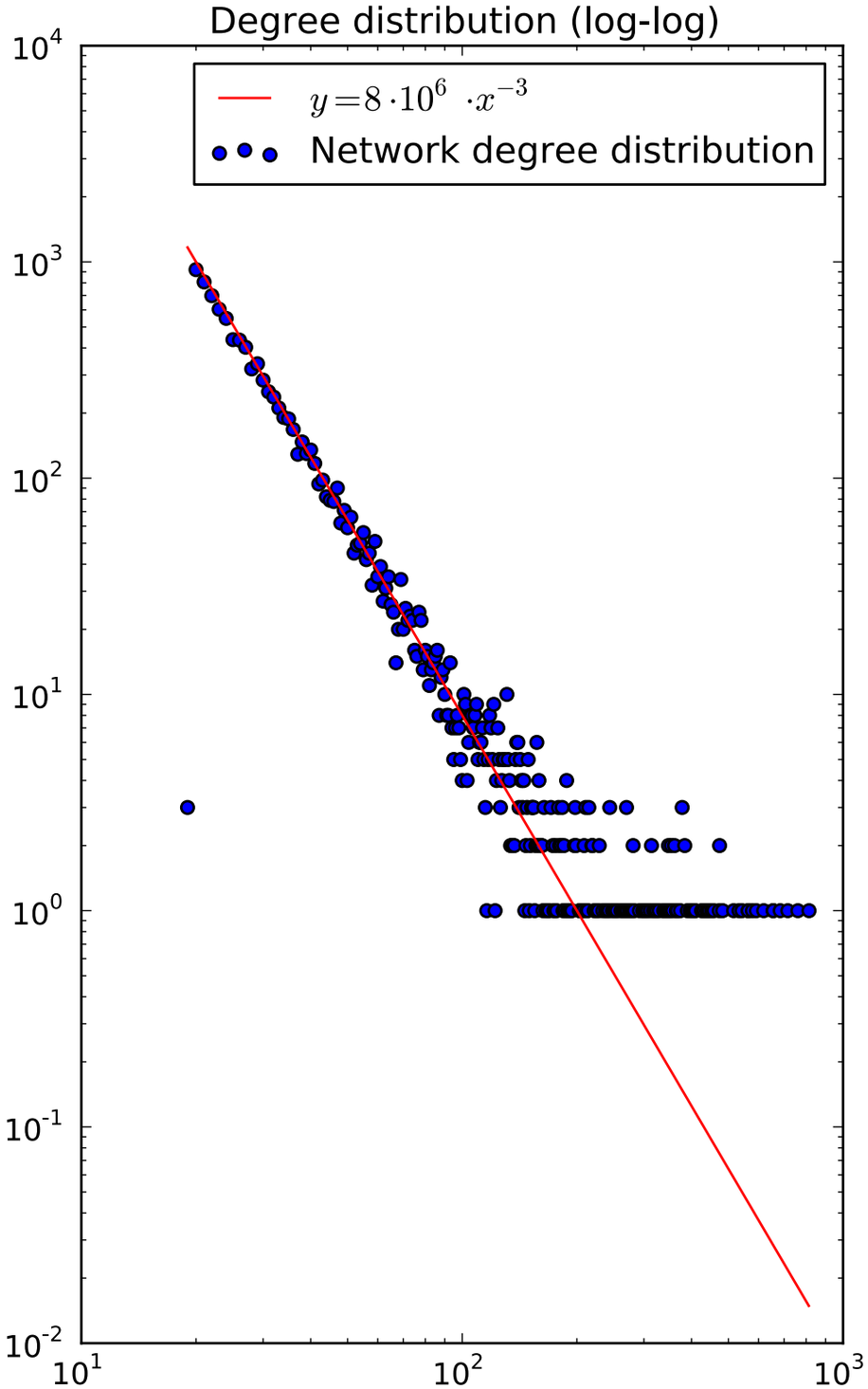
$$p_k / p_{k'} = p_{2k} / p_{2k'}$$

- Not all moments are defined!

$$E[x^n] = \int x^n p(x)\,dx =$$

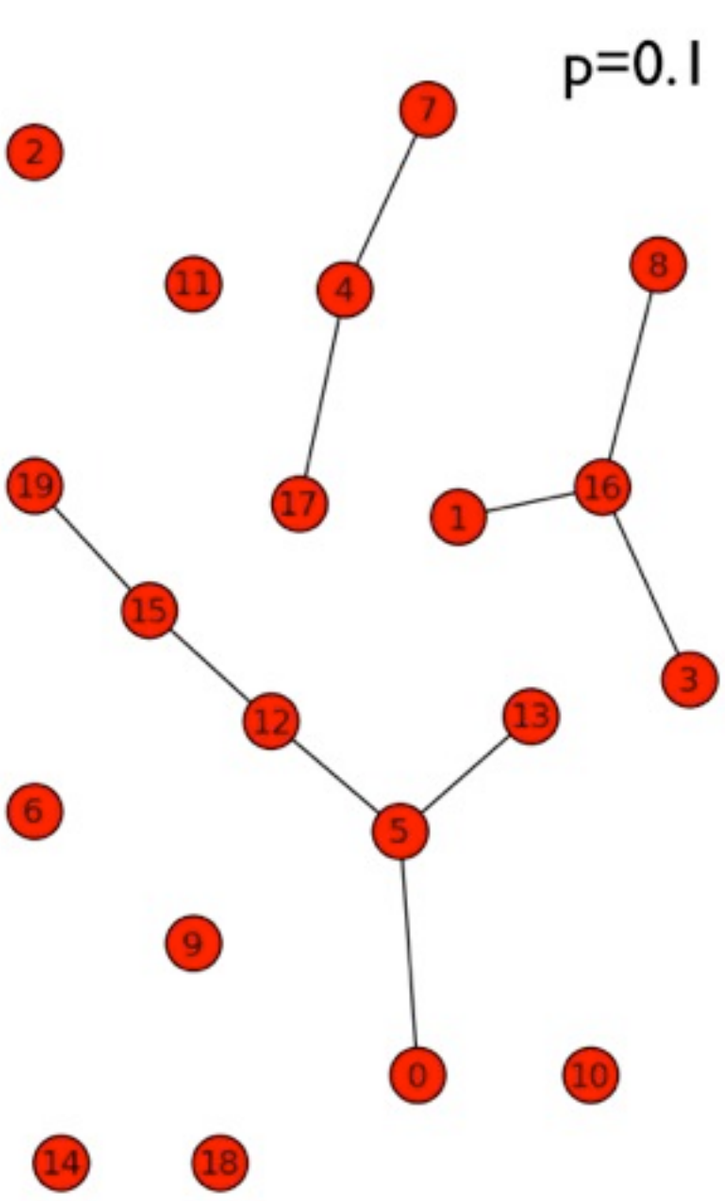
$$= \int x^{n-\alpha}\,dx$$

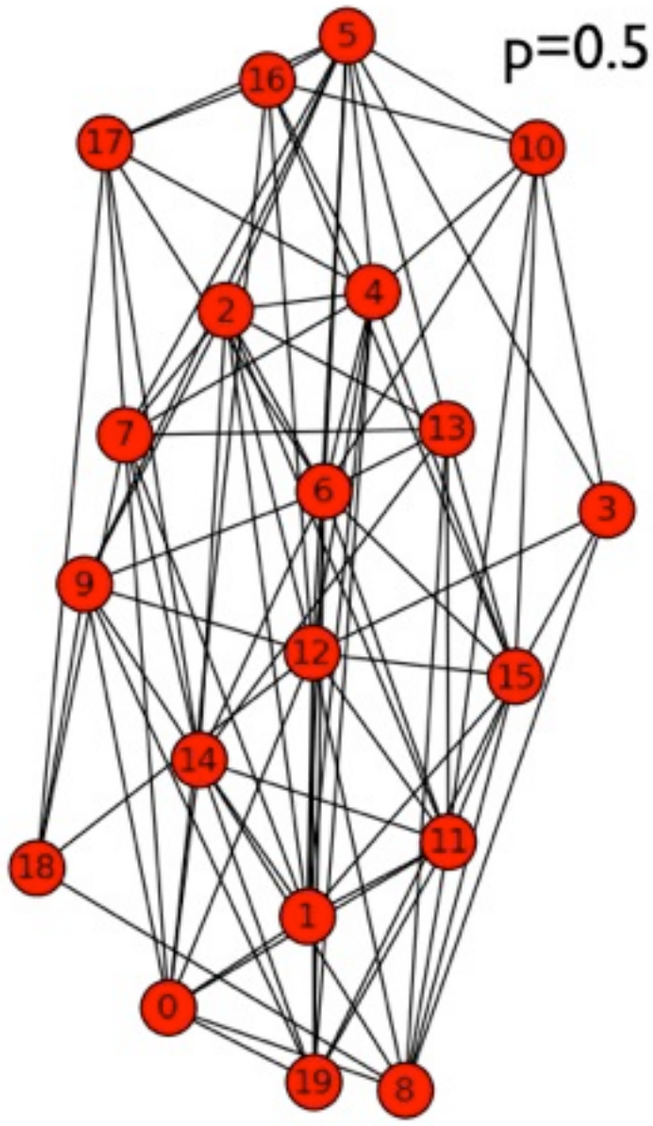

Degree distribution
$y = 8 \cdot 10^6 \cdot x^{-3}$
Network degree distribution
1200
1000
800
600
400
200
0
−200
0
200
400
600
800

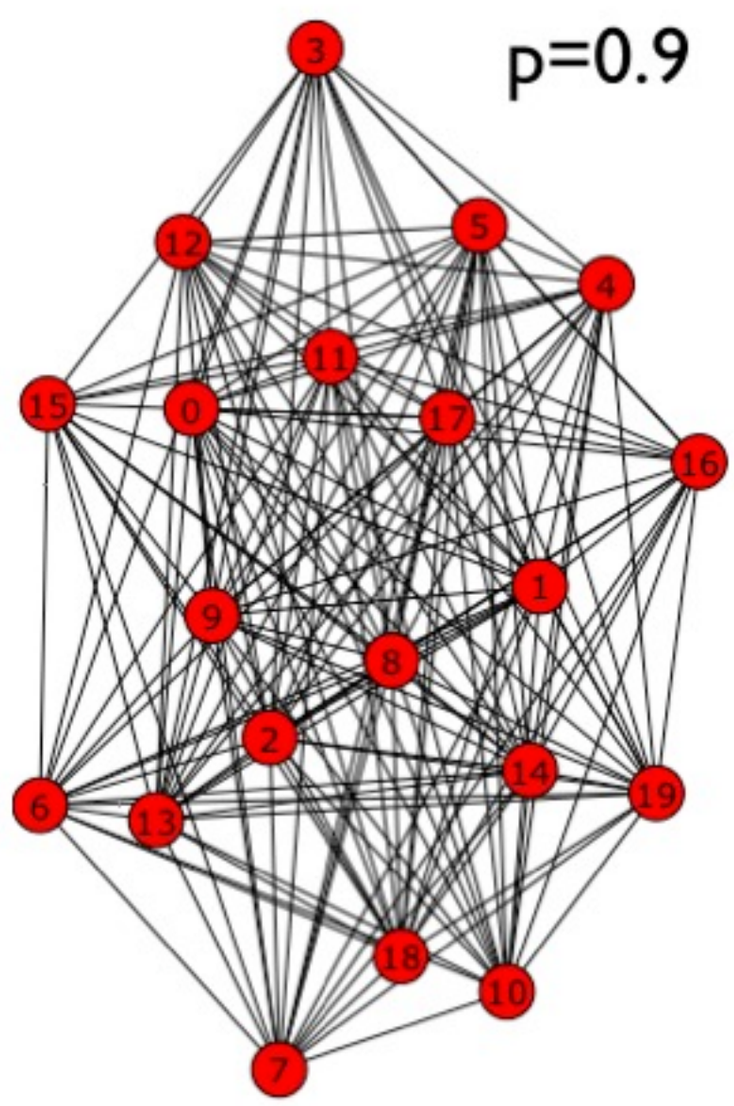

Degree distribution (log-log)
$y = 8 \cdot 10^6 \cdot x^{-3}$
Network degree distribution
10^4
10^3
10^2
10^1
10^0
10^{-1}
10^{-2}
10^1
10^2
10^3

Random Graphs

- An Erdös-Rényi random graph model G(n, m) is a probability distribution over the set of simple graphs with n nodes and m edges
- A mathematically equivalent model (for large n) is G(n, p). When a graph is drawn from G(n, p) each possible edge is independently placed with probability p
- Other than the naive ways to code the process, there are efficient O(n+m) algorithms (implemented in networkx)







Random Graphs

ER-Random Graphs

- It is unsurprising that ER-random graphs are not good models for social networks (though studies on a high school romance network shows striking similarities with ER-random graphs

Average degree: $\bar{k} = (1-p)n$

Clustering coefficient: $C = \bar{k}/(n-1)$

Degree distribution: $p_k \simeq e^{-c}\frac{\bar{k}^k}{k!}$

Diameter in the order of log *n*

Social Networks and Random Graphs

- Social Networks have short characteristic path length (in the order of log *n*)
- Social Networks have high clustering coefficient (wrt. Random Graphs with comparable number of nodes and average degree)
- Social Networks have right skewed degree distributions
- Generative approach?

Real World Examples

OSN	Users	Links	$\langle k_i \rangle$	C	CPL	γ	assort.
Club Nexus [1]	2.5 K	10 K	8.2	0.17	4	-	-
Cyworld [2]	12 M	191 M	31.6	0.16	3.2	4; 1	-0.13
Cyworld T. [2]	92 K	0.7 M	15.3	0.32	7.2	-	0.43
Orkut [2]	100 K	1.5 M	30.2	0.30	3.8	3.7	0.31
Orkut [3]	3 M	223 M	106	0.17	4.2	1.50	0.07
Flickr [3]	1,8 M	22 M	12.2	0.31	5.7	1.7	0.20
Live Journal [3]	5 M	77 M	17	0.33	5.9	1.6	0.18
Youtube [3]	1,1 M	5 M	4.29	0.14	5.1	~ 2	-0.03

- **blue bold**: more than 90% of nodes analysed
- black: less than 90%, more than 1% of nodes analysed
- *red italic*: $<$ 1% of nodes analysed

Group Level Properties

- Identification of cohesive sub-groups
  - one-mode networks
    (n-clique, n-clan, n-club, k-plex, k-core, LS set)
  - two-mode networks
- Network Positions
- Blockmodels
- Networkx gives them all!
  - Efficiency, interpretation

Group Level Properties

- Identification of cohesive sub-groups
  - one-mode networks
    (n-clique, n-clan, n-club, k-plex, k-core, LS set)
  - two-mode networks
- Network Positions
- Blockmodels
- Networkx gives them all!
  - Efficiency, interpretation

- Highly connected core
- Fringe
- Stars & Isolated Nodes

Node level properties

- "Centrality" metrics
- Ranking
- Study distribution & correlation

Betweenness centrality

- Let $P_i(k,j)$ be the number of shortest paths between k and j that i lies on
- Let $P(k,j)$ the total number of shortest paths between k and j
- If $P_i(k,j)P(k,j)^{-1} \cong I$, then i lies on most shortest paths between k and j
- The betweenness centrality of a node i is:

$$c_i^B = \sum_{\substack{k \neq j \\ i \notin \{k,j\}}} \binom{n-1}{2}^{-1} \frac{P_i(k,j)}{P(k,j)}$$

```
def attack(graph, centrality_metric):
    graph = graph.copy()
    steps = 0
    ranks = centrality_metric(graph)
    nodes = sorted(graph.nodes(), key=lambda n: ranks[n])

    while nx.is_connected(graph):
        graph.remove_node(nodes.pop())
        steps += 1
    else:
        return steps
```

```
Creating powerlaw cluster with 1000 elements.
Creating G(1000,0.007964)
Starting attacks.
Social network broke after 220 steps with random
attack.
Random network broke after 10 steps with random
attack.
Social network broke after 22 steps with betweenness
ranking.
Random network broke after 157 steps with betweenness
ranking.
Social network broke after 19 steps with pagerank.
Random network broke after 149 steps with pagerank.
Social network broke after 19 steps with degree.
Random network broke after 265 steps with degree.
```

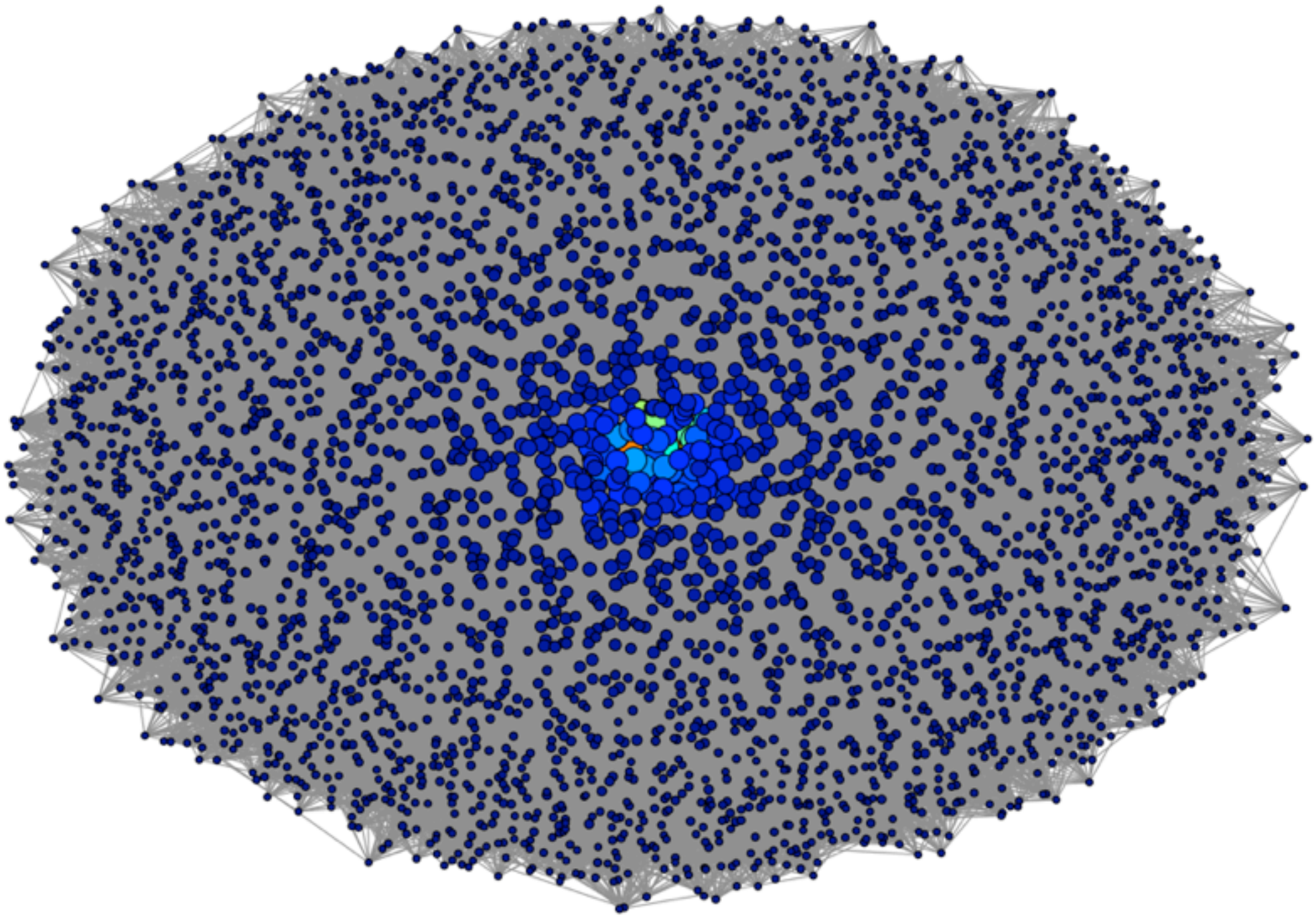
Visualization

Visualization

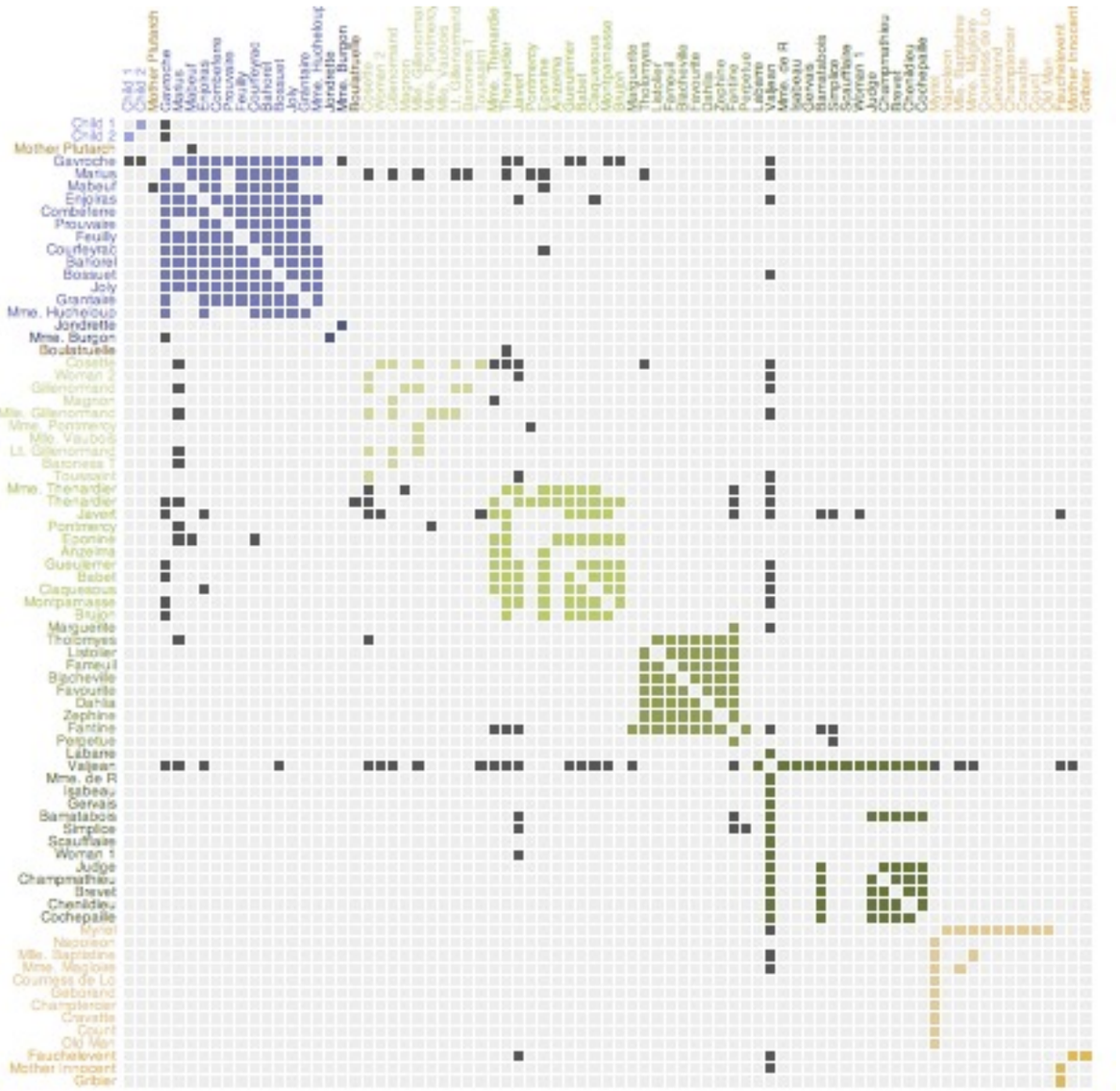
- Networkx
  - Matplotlib
  - PyGraphViz
  - Pydot
- Gephi
- Guess
- Protovis

- Force directed algorithms
- Energy minimization
- Fixed layouts (circle)
- Different colors on nodes
- Dynamic Manipulation
- ?

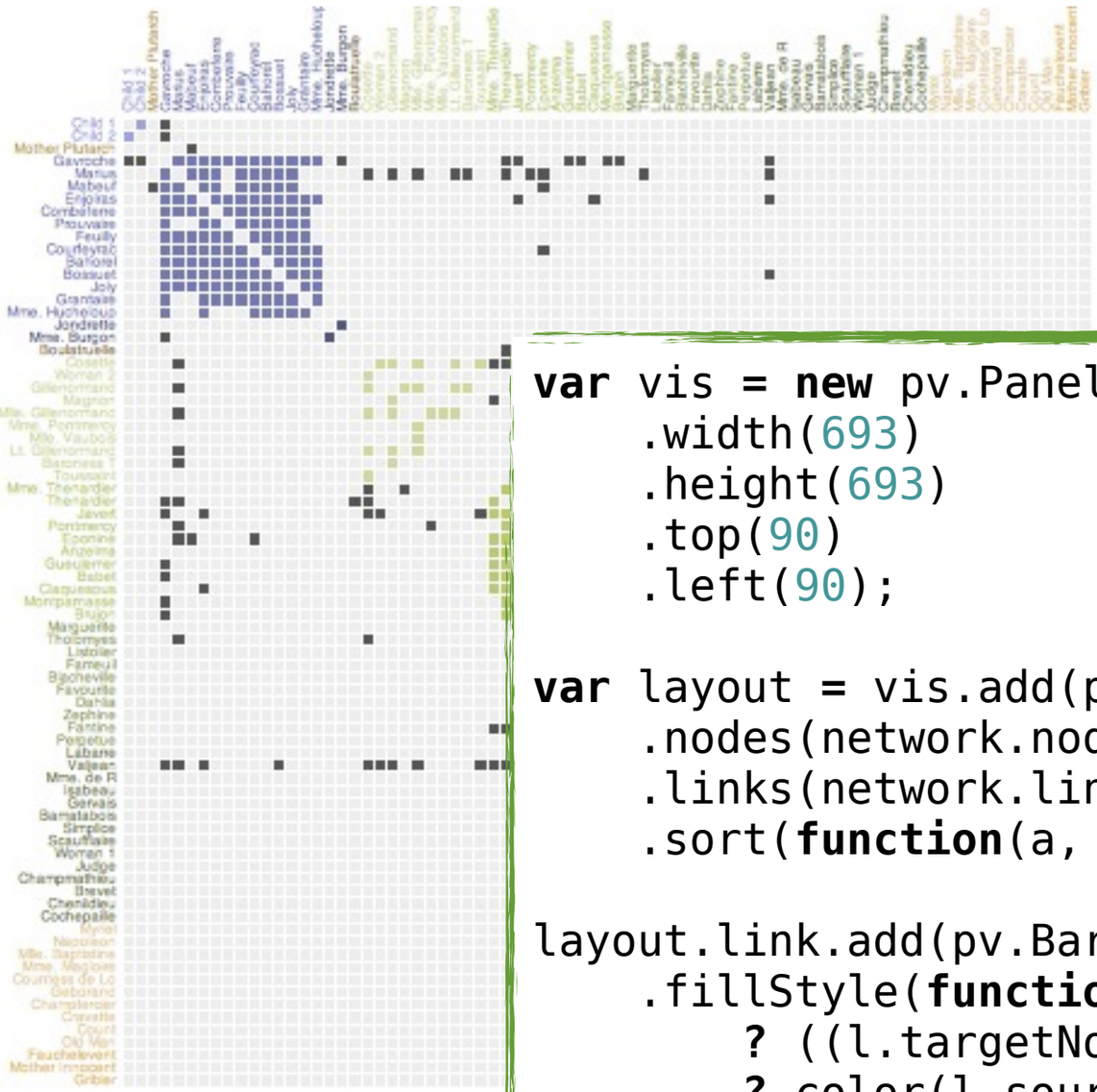
```
nx.draw_graphviz(g, node_size=nx.degree(g).values(), with_labels=False,
                    node_color=nx.betweenness_centrality(g).values(),
                    edge_color='grey')
```



```
import json
from random import randint
import networkx as nx

graph = nx.powerlaw_cluster_graph(1000, 4, 0.05)
dict_of_lists = nx.to_dict_of_lists(graph)

nodes = [dict(nodeName=str(node), group=randint(1, 100))
         for node in dict_of_lists.iterkeys()]
edges = []
for node, neighbors in dict_of_lists.iteritems():
    for neighbor in neighbors:
        edges.append(
            dict(source=node, target=neighbor, value=1)
        )
        edges.append(
            dict(target=node, source=neighbor, value=1)
        )
json_like_structure = dict(nodes=nodes, links=edges)
with open('social_network.js', 'w') as fp:
    json.dump(json_like_structure, fp)
```

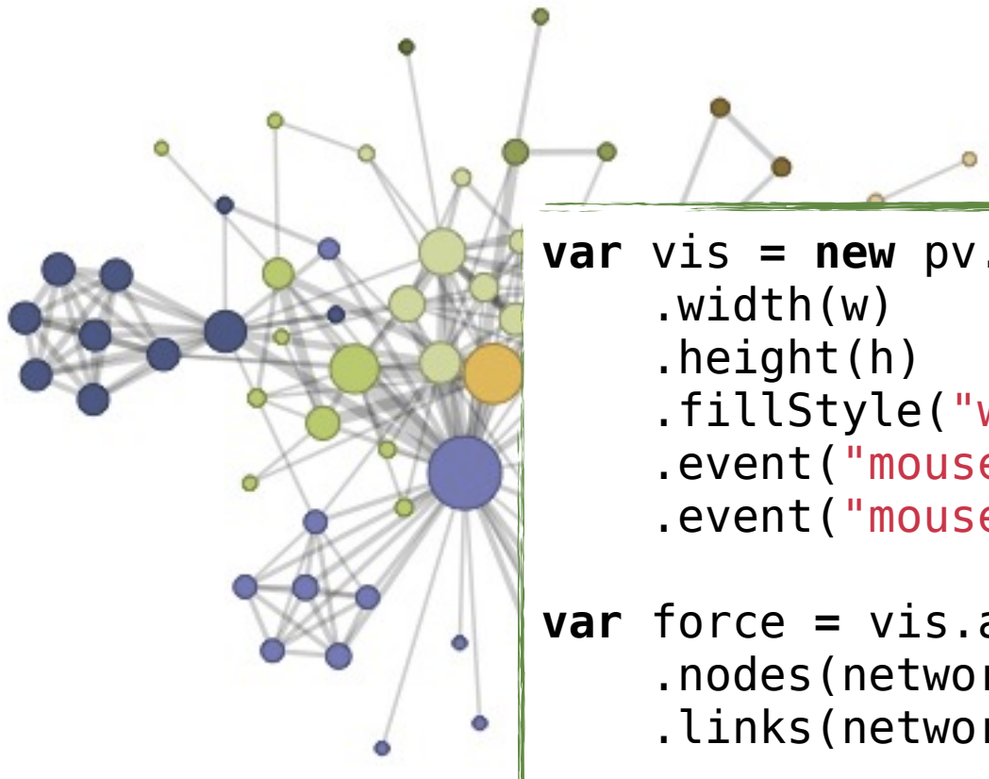

Child 1
Child 2
Mother Plutarch
Gavroche
Marius
Mabeuf
Enjolras
Combeferre
Prouvaire
Feuilly
Courfeyrac
Bahorel
Bossuet
Joly
Grantaire
Mme. Hucheloup
Jondrette
Mme. Burgon
Boulatruelle
Cosette
Woman 2
Gillenormand
Magnon
Mlle. Gillenormand
Mme. Pontmercy
Mlle. Vaubois
Lt. Gillenormand
Baroness T
Toussaint
Mme. Thenardier
Thenardier
Javert
Pontmercy
Eponine
Anzelma
Gueulemer
Babet
Claquesous
Montparnasse
Brujon
Marguerite
Tholomyes
Listolier
Fameuil
Blacheville
Favourite
Dahlia
Zephine
Fantine
Perpetue
Labarre
Valjean
Mme. de R
Isabeau
Gervais
Barnatabois
Simplice
Scaufflaire
Woman 1
Judge
Champmathieu
Brevet
Chenildieu
Cochepaille
Myriel
Napoleon
Mlle. Baptistine
Mme. Magloire
Countess de Lo
Geborand
Champtercier
Cravatte
Count
Old Man
Fauchelevent
Mother Innocent
Gribier
10

Protovis



```
var vis = new pv.Panel()
    .width(693)
    .height(693)
    .top(90)
    .left(90);

var layout = vis.add(pv.Layout.Matrix)
    .nodes(network.nodes)
    .links(network.links)
    .sort(function(a, b) b.group - a.group);

layout.link.add(pv.Bar)
    .fillStyle(function(l) l.linkValue
        ? ((l.targetNode.group == l.sourceNode.group)
        ? color(l.sourceNode) : "#555") : "#eee")
    .antialias(false)
    .lineWidth(1);

layout.label.add(pv.Label)
    .textStyle(color);

vis.render();
```

Protovis



```
var vis = new pv.Panel()
    .width(w)
    .height(h)
    .fillStyle("white")
    .event("mousedown", pv.Behavior.pan())
    .event("mousewheel", pv.Behavior.zoom());

var force = vis.add(pv.Layout.Force)
    .nodes(network.nodes)
    .links(network.links);

force.link.add(pv.Line);

force.node.add(pv.Dot)
    .size(function(d) (d.linkDegree + 4) * Math.pow
(this.scale, -1.5))
    .fillStyle(function(d) d.fix ? "brown" : colors
(d.group))
    .strokeStyle(function() this.fillStyle().darker())
    .lineWidth(1)
    .title(function(d) d.nodeName)
    .event("mousedown", pv.Behavior.drag())
    .event("drag", force);

vis.render();
```

Protovis

Thanks for your Kind Attention

Enrico Franchi (efranchi@ce.unipr.it)

https://gist.github.com/1010039

PageRank

Page Rank

- Being cited by an (important) page which collects links is not the same than being cited by a page "on the same subject"
- Important pages are cited more often

$$x_i = \sum_j A_{ji} \frac{x_j}{k_j}$$

Page Rank

- In order to simplify the notation, we define the H matrix: $H_{ij} = A_{ij} k_j^{-1}$
- We can try to compute $\mathbf{xH} = \mathbf{x}$ with successive approximations, like in $\mathbf{x}(t) = \mathbf{x}(0)\mathbf{H}^t$ with $t \to \infty$
- Each iteration takes $O(n^2)$ operations
  - the number of non-zero entries is O(n), which makes the computation O(n)
- Convergence?

Interpretation of Page Rank

- Random Surfer
- If time spent surfing approximates infinity, time spent on a given page is a measure of that page importance
- Dangling Nodes

Perron-Froebenius Theorem

- If T is a nonnegative row-stochastic matrix (i.e., the entries in each row sum to 1), there is a non negative eigenvector **v** such that $\mathbf{vT} = \lambda\mathbf{v}$ and has a corresponding eigenvalue $\lambda=1$
- If T^t has all positive entries for some t (i.e., T is primitive), then all other eigenvalues have magnitude less than 1
  - A matrix is primitive if it has only one eigenvalue on the spectral circle

Primitivity Adjustment

- The **H** matrix has almost all the right properties. Dangling nodes make it non-stochastic (we say it's quasi-stochastic)
- With the random walker intuition, we can fix everything

$$\mathbf{S} = \mathbf{H} + \mathbf{a}\left(\frac{1}{n}\mathbf{e}^T\right)$$

- where a is the *dangling node vector* (a_i=1 if i is a dangling node)

Markov Chains interpretation

- **S** is the matrix of a Markov process
- It is stochastic, irreducible (equivalent to say that the corresponding graph is strongly connected) and aperiodic
  - aperiodic + irreducible → primitive
- From a mathematical point of view, everything is fine. However, we are implying that surfers never "jump" to entirely new pages

The Google Matrix

- Let α be a scalar between 0 and 1

$$\mathbf{G} = \alpha\mathbf{S} + (1-\alpha)\frac{\mathbf{e}\mathbf{e}^T}{n}$$

- G is stochastic, because the convex combination of two stochastic matrices is stochastic
- G is irreducible (every page is connected with every other page)
- G is aperiodic
- G is (unfortunately) dense

Computing the PageRank

$$\mathbf{G} = \alpha\mathbf{S} + (1-\alpha)1/n\,\mathbf{e}\mathbf{e}^T$$
$$= \alpha(\mathbf{H} + 1/n\,\mathbf{a}\mathbf{e}^T) + (1-\alpha)1/n\,\mathbf{e}\mathbf{e}^T$$
$$= \alpha\mathbf{H} + (\alpha\mathbf{a} + (1-\alpha)\mathbf{e})1/n\,\mathbf{e}^T$$

- We could see the computation as:
  - an eigenvector problem:

    $\mathbf{x}^T = \mathbf{x}^T\mathbf{G}$

    $\mathbf{x}^T\mathbf{e} = 1$
  - solution of linear hom. system

    $\mathbf{x}^T(\mathbf{I} - \mathbf{G}) = \mathbf{0}^T$

    $\mathbf{x}^T\mathbf{e} = 1$

The Power-Method

$$\begin{aligned}
\mathbf{x}^{(k+1)T} &= \mathbf{x}^{(k)T}\mathbf{G} \\
&= \alpha\mathbf{x}^{(k)T}\mathbf{S} + (1-\alpha)\,1/n\,\mathbf{x}^{(k)T}\mathbf{e}\mathbf{e}^{T} \\
&= \alpha\mathbf{x}^{(k)T}(\mathbf{H} + 1/n\,\mathbf{a}\mathbf{e}^{T}) + (1-\alpha)\,1/n\,\mathbf{x}^{(k)T}\mathbf{e}\mathbf{e}^{T} \\
&= \alpha\mathbf{x}^{(k)T}\mathbf{H} + (\alpha\mathbf{x}^{(k)T}\mathbf{a} + (1-\alpha))\,\mathbf{e}^{T}/n
\end{aligned}$$

- The power method is usually slow, but has lots of nice properties:
  - is matrix-free (matrix is only accessed, not manipulated)
  - the matrix is easy to distribute, since its sparse

Personalization vector

- Instead of assuming a random probability to jump on any page, we consider an "personalized probability"

$$\begin{aligned}
\mathbf{x}^{(k+1)T} &= \mathbf{x}^{(k)T}\mathbf{G}_{\mathbf{v}} \\
&= \alpha\mathbf{x}^{(k)T}\mathbf{S}_{\mathbf{v}} + (1-\alpha)\mathbf{x}^{(k)T}\mathbf{e}\mathbf{v}^{T} \\
&= \alpha\mathbf{x}^{(k)T}(\mathbf{H} + \mathbf{a}\mathbf{v}^{T}) + (1-\alpha)\mathbf{x}^{(k)T}\mathbf{e}\mathbf{v}^{T} \\
&= \alpha\mathbf{x}^{(k)T}\mathbf{H} + (\alpha\mathbf{x}^{(k)T}\mathbf{a} + (1-\alpha))\mathbf{v}^{T}
\end{aligned}$$

... in Python

- Use networkx
  - nx.pagerank
  - nx.pagerank_numpy
  - nx.pagerank_scipy